From Growth Curves to Mixed(Random)-Effects Models

lm vs lme(r)

$$\eta_p(t) = \eta_p(0) + \theta_p t$$

Data $y_{pj} = \eta_p(t_{pj}) + \varepsilon_{pj}$

ordered times $j = 1 \ldots T_p$ # obs for p (missing or not)

intercepts and slopes may differ across $p = 1, \ldots, N$

No W

$E_p(\theta_p) = \mu_\theta$

$E_p(\eta_p(0)) = \mu_{\eta(0)}$

Random
$\eta_p(0) - \mu_{\eta(0)}$
$\theta_p - \mu_\theta$

Fixed
$\mu_{\eta(0)}$
μ_θ

Mixed Effects Model

$$\underset{\sum_{p=1}^{N} T_p \times 1}{y} = \underset{\sum T_p \times 2}{X} \underset{2 \times 1}{\beta} + \underset{\sum T_p \times 2N}{Z} \underset{2N \times 1}{\gamma} + \varepsilon$$

fixed
ave. growth curve

$$X = \begin{bmatrix} 1 & t_{p1} \\ \vdots & \vdots \\ 1 & t_{pT} \end{bmatrix} \qquad \beta = \begin{bmatrix} \mu_{\eta(0)} \\ \mu_\theta \end{bmatrix}$$

random

$$\gamma = \begin{bmatrix} \eta_p(0) - \mu_{\eta(0)} \\ \theta_p - \mu_\theta \end{bmatrix}$$

2 rows each p

blocks

$$Z = \begin{bmatrix} \begin{bmatrix} 1 & t_{p1} \\ \vdots & \vdots \\ 1 & t_{pT} \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} \quad & \quad \end{bmatrix} \end{bmatrix}$$

block elements of G

$$\begin{bmatrix} \sigma^2_{\eta(0)} & \sigma_{\eta(0)\theta} \\ & \sigma^2_\theta \end{bmatrix}$$

$$Var(Y) = V = Z G Z' + R$$

also D aka

with W

Fixed

$$E(\eta(t)|w) = \mu_{\eta(t)} + \beta_{\eta(t)w}(w - \mu_w)$$

$$E(\theta|w) = \mu_\theta + \beta_{\theta w}(w - \mu_w)$$

Random

$$\eta_p(t) - E(\eta(t)|w)$$

$$\theta_p - E(\theta|w)$$

$$Y = X\beta + Z\gamma + \varepsilon$$

expanding

$$X_{\sum T_p \times 4} = \begin{bmatrix} 1 & t_{p1} & w_p & w_p * t_{p1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_{pT_p} & w_p & w_p * t_{pT_p} \end{bmatrix}$$

fixed

$$\beta_{4\times 1} = \begin{bmatrix} \mu_{\eta(t)} - \beta_{\eta(t)w}\mu_w \\ \mu_\theta - \beta_{\theta w}\mu_w \\ \beta_{\eta(t)w} \\ \beta_{\theta w} \end{bmatrix} \begin{matrix} \text{int} \\ \text{time} \\ w \\ \text{time} * w \end{matrix}$$

random

$$\gamma_{2N\times 1} = \begin{bmatrix} \eta_p(t) - E(\eta(t)|w_p) \\ \theta_p - E(\theta|w_p) \\ \vdots \end{bmatrix}$$

G-matrix contains conditional variances

$Var(\theta|w)$ $Var(\eta(t)|w)$

See NCFem, Frames 7, 8 etc